Abstract

The four-color problem has remained unsolved (except as aided by computing machines) for some 171 years since it was first posed in 1854, having the following essential elements:

1. a *country* is a region (an open, connected point set) in the plane;
2. a *map* m is a collection of non overlapping countries where $\bar{m}$ is connected;
3. [map m includes countries a, b; the closures of a, b have a point in common; $B_{a,b}$ denotes the boundary shared between countries a, b; α is an arc; countries a, b *are adjacent to each other*] $\Leftrightarrow$ [$\alpha \subseteq B_{a,b}$]; and
4. no two countries in map m having the same color are *adjacent to each other*.

The approach taken herein toward solving the four-color problem begins (after the present abstract) with a literature-review section and a section formally defining *a country, a map, the adjacency of countries, the mutual separation of countries, a corner of the countries in a map*, *an isolated point in the boundary of a map*, and the *four-colorability of a map.* The present paper continues with the statements of three lemmas (one concerning the concept of *adjacency* of countries, another concerning the concept of *mutual separation* of countries, and a third concerning the concept of a *corner of the countries in a map.* A theorem stating that every map is four-colorable follows, and relies upon, the three lemmas. A section entitled Crucial Definitions and one entitled Set-Builder Notation precede a discussion section. The paper concludes with a section of references.

The Four-Color Problem

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1 Literature Review

Francis Guthrie has been credited with originating the four-color problem in 1852.[1]

Augustus DeMorgan has been credited with first observing in 1852[6] that four mutually adjacent countries require that "one or more of them be inclosed by the rest".

In his 1860 letter to Sir William Rowan Hamilton[7][2]—noting that, the first three illustrations included herewith have every degree of fidelity with DeMorgan's originals, and that the fourth one illustrates the idea of mutual separation in a map having five countries—DeMorgan wrote:

> "My Dear Hamilton
>
> "A student of mine asked me to day to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary line are differently coloured — four colours may be wanted but not more — the following is his case in which four are wanted
>
> "A B C D are names of colours
>
> "Query cannot a necessity for five or more be invented As far as I see at this moment, if four alternate compartments have each boundary line in common with one of the others, four of them inclose the fourth, and prevent any fifth from communion with it. If this be

[1] "Tinting Maps.—In tinting maps, it is desirable for the sake of distinctness to use as few colours as possible, and at the same time no two coterminous divisions ought to be tinted the same. Now, I have found by experience that four colours are necessary and sufficient for this purpose,—but I cannot prove that this is the case, unless the whole number of divisions does not exceed five. I should like to see (or know where I can find) a general proof of this apparently simple proposition, which I am surprised never to have met with in any mathematical work. F. G."[8] Also see page 37 in the cited book:[14] "It was not until 1959 that the geometer H.S.M. Coxeter set the story straight, and since then Francis Guthrie has been universally recognized as the true originator of the four-color problem."

[2] In 1860, DeMorgan wrote,[4, 7] "This arises in the following way. We never need four colours in a neighborhood unless there be four counties, each of which has boundary lines in common with each of the other four. Such a thing cannot happen with four areas unless one or more of them be inclosed by the rest; and the colour used for the inclosed county is thus set free to go on with. Now this principle, that four areas cannot each have common boundary with all the other four without inclosure, is not, we fully believe, capable of demonstration upon anything more evident and more elementary; it must stand as a postulate."

true, four colours will colour any possible map without any necessity for colour meeting colour except at a point.

"Now it does seem that drawing four compartments with common boundary A B C two and two — you cannot make a fourth take boundary from all, except by inclosing one — But it is tricky work and I am not sure of all convolutions — What do you say? And has it, if true been noticed? My pupil says he noticed it in colouring a map of England.

"B is inclosed

"The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did If this rule be true the following proposition of logic follows If A B C D be four names of which say two might be confounded by breaking down some wall of definitions, then some one of the names must be a species of some name which includes nothing external to the other four

"Yours truly,

"DeMorgan

"Oct 23/52."

DeMorgan has been credited with first observing in 1860 that five mutually adjacent countries is impossible.[4, 7, 14][3]

In 1890, P. J. Haewood proved the Five Colour Theorem using graph theory, demonstrating that every five-country map is four-colorable.[10]

Georg Cantor proved important theorems concerning transfinite set theory (1878, 1883), as well as offering a diagonal enumeration argument (1891), which were necessary to prove his theorems related to ordinal numbers and cardinal numbers. Cantor also established the important notion of "one-to-one correspondence" (1874). (See Philip Jourdain (ed., 1915), English translation of Cantor's "Contributions to the Founding of the Theory of *transfinite numbers*".[5])

Kurt Gödel proved two theorems: (1) The Completeness Theorem (1930) and (2) The Incompleteness Theorem (1931). The webpage "Kurt Gödel" gives a full treatment of Gödel's mathematics along with his biography.[12] Transferring his affiliation from the university of Vienna to Princeton's Institute for Advanced Studies in 1940, Gödel became strong friends with Albert Einstein, also at the Institute for Advanced Studies until Einstein's death in 1955.

Kenneth Appel and Wolfgang Haken presented a solution for the four-color problem (or the map-coloring problem) in their papers of 1977 and 1989 with reliance on computer code and computing machines.[1, 3]

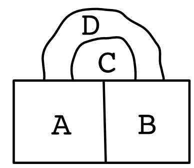
In 2008, Georges Gonthier formalized and proved the four-color theorem through the use of computing machines and general purpose theorem-proving computer code, invoking the idea of a *corner of the countries in a map.*[9]

2 Definitions

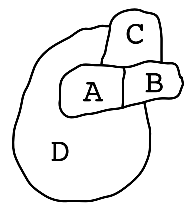
Def. 2.1. *Country: A* country *is a region* (*an open, connected point set*) *in the plane.*

[3]See page 108 in the cited reference[2], here quoted: "DeMorgan proved that it is not possible for five countries to be in a position such that each of them is adjacent to the other four", a principle known elsewhere as "the Separation Axiom".

DeMorgan's "Figure 1"



DeMorgan's "Figure 2"



DeMorgan's "Figure 3"

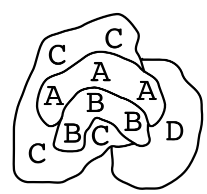
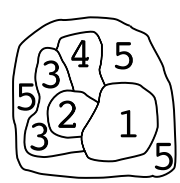


Illustration of *mutual separation* for a five-country map. The country colored 2 is *mutually separated from* the country colored 5, since the closures of the countries colored 2 and 5 have no point in common (and are subject to the Separation Axiom.[4, 7, 14])



Def. 2.2. *Map: A* map *m is a collection of non overlapping countries where $\bar{m}$ is connected. Set of all maps: $M = \{m$ is a* map$\}$.

Def. 2.3. *Adjacency:* $[\forall$*map* $m \in M, |m| > 1, \exists$*countries* $a, b \in m \ni \bar{a} \cap \bar{b} =$ *arc* $\alpha] \Leftrightarrow [a, b$ are adjacent to each other$]$.

Def. 2.4. *Mutual Separation of Countries:* $[\forall m \in M, |m| > 1, \exists$*countries* $a, b \in m, \bar{a} \cap \bar{b} = \emptyset] \Leftrightarrow [$*countries* a, b are mutually separated$]$.[4]

Def. 2.5. *Point P is a* corner *of map m:* $[\forall$*map* $m \in M, |m| > 2, \forall$*country* $t \in m$, *where* $B_t =$ *the outer boundary of country* $t, \exists$*point* $P \ni P \in B_t] \Leftrightarrow [P$ *is a* corner *of map* $m]$.[5]

Def. 2.6. *Isolated Point:* $[$*map* $m \in M, B$ *is the boundary of map* m, P *is a non limit point of* $B] \Leftrightarrow [P$ *is an* isolated point *of* $B]$.[6]

Def. 2.7. *The set of all four-colorable maps*: $F = \{m \in M$ *is* four-colorable$\}$.

1. $\big[\forall m \ni |m| > 1, G$ *is a collection of ordered pairs of countries* $(u, v) \ni [u, v \in m] \ni |G| = n$, *where n does not exclude any possible transfinite number,*[5, 12] *and*
2. $\forall \eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, *and each member of at least one ordered pair* $(a, b) \in h_\eta$ *is* mutually adjacent to *its pair-mate, and each member of every **remaining** ordered pair* $(a, b) \in G$ *is* mutually separated from *its pair-mate*$]\big] \Leftrightarrow [$*map* $m \in F]$.

3 Three Lemmas

Lem. 3.1. $[\forall$*map* $m \in M, |m| > 1, \exists$*countries* $a, b \in m \ni \bar{a} \cap \bar{b} =$ *arc* $\alpha] \Leftrightarrow [a, b$ are adjacent to each other$]$.

Proof. **Consider** the logical negation of the $\Leftarrow$ case of Lem. 3.1: $[\forall$map $m \in M, |m| > 1, \exists$countries $a, b \in m \ni \bar{a} \cap \bar{b} =$ arc $\alpha] \not\Leftarrow [a, b$ *are adjacent to each other*$]$. (Premise of the $\Leftarrow$ case of the present lemma)

Then "$\neg[\forall$map $m \in M, |m| > 1, \exists$countries $a, b \in m$, arc $\alpha = \bar{a} \cap \bar{b}]$", *i.e.*, either (a) "$[\nexists$map $m \in M, |m| > 1, \forall$countries $a, b \in m$, arc $\alpha = \bar{a} \cap \bar{b}]$" or (b) "$[\exists$map $m \in M, |m| \leq 1, \forall$countries $a, b \in m$, arc $\alpha = \bar{a} \cap \bar{b}]$" or (c) "$[\exists$map $m \in M, |m| > 1, \forall$countries $a, b \in m$, arc $\alpha \neq \bar{a} \cap \bar{b}]$." (a) is impossible because then map m does not exist (Contradiction); (b) is false because then two

[4] It may be observed that specifications such as "$m \in M, |m| > 1$" invoke the concept of "cardinality"—never regarded as anything other than a number (albeit, perhaps, a **"transfinite" number**): the cardinality of every map in the present paper has been specified **as a number**. Thus, the approach taken herein to solving the four-color problem allows for large cardinals.[5, 12]

[5] Def. 2.5 defines the *corner* of map m as a point P that is common to the outer boundaries of a collection B_t of three or more countries belonging to m. In general, the cardinality of B_t will be a large cardinal. (See Footnote 4.)

[6] The concept of *isolated point* (see Def. 2.6) is inconsistent with (at least some of) the published diagrams of Francis Guthrie (who is credited with originating the Four-Color Problem[14]), and those of DeMorgan (with whom Guthrie corresponded) (see the quoted letter, below)—illustrating that neither of them accepted any *isolated point* in any map's boundary.[6, 8] In his letter of 23 Oct. 1852 to William Rowan Hamilton, DeMorgan stated that, "... four colours will colour any possible map without any necessity for colour meeting colour except at a point". If the number of *isolated points* were finite, the problem perhaps would present minimal difficulties. But if there were infinitely many or uncountably many, where such points might have any configuration (fractal-like, for example), doubtless the difficulties would be daunting. *Isolated points* therefore will not be considered further in this paper.

countries do not exist in m (Contradiction); and (c) is false since $[\bar{a} \cap \bar{b}]$ includes an *isolated point* P, which violates Footnote 2.6 (Contradiction).

Hence, a contradiction having been reached for every possibility under the $\Leftarrow$ case of the present lemma, the above argument is complete.

Consider now the statement "$[\forall$map $m \in M \ni |m| > 1, \exists$countries $a, b \in m \ni \bar{a} \cap \bar{b} =$ arc $\alpha] \Rightarrow [a, b$ *are adjacent to each other*$]$", which, by Def. 2.3, satisfies the $\Rightarrow$ case of the present lemma. (Premise of the $\Rightarrow$ case of the present lemma)

Hence, since both the $\Leftarrow$ case and the $\Rightarrow$ case hold, the present lemma holds, and establishes an existence proof for the concept of *adjacency*. $\square$

Lem. 3.2. $[\forall m \in M, |m| > 1, \bar{a} \cap \bar{b} = \emptyset] \Leftrightarrow [$*countries* a, b are mutually separated$]$.

Proof. **Consider** the logical negation of the $\Leftarrow$ case of the present lemma: $[\forall m \in M, |m| > 1, \bar{a} \cap \bar{b} = \emptyset] \not\Leftarrow [$countries a, b *are mutually separated*$]$.

Then "$\neg[\forall m \in M, |m| > 1, \bar{a} \cap \bar{b} = \emptyset]$", *i.e.*,

either (a) "$[\exists m \in M, |m| > 1, \bar{a} \cap \bar{b} \neq \emptyset]$" (so that

either (1) "$[\exists m \in M, |m| > 1, \bar{a} \cap \bar{b} \neq \emptyset]$", implying the existence of an isolated point, contrary to Footnote6, or (2) "$[\exists m \in M, |m| > 1, \bar{a} \cap \bar{b} = \emptyset]$", implying "$[\exists$arc $\alpha \ni \alpha = \bar{a} \cap \bar{b}]$" and hence satisfying the both the $\Rightarrow$ and the $\Leftarrow$ case of Lem. 3.2);

or (b) "$[\nexists m \in M, |m| > 1, \bar{a} \cap \bar{b} = \emptyset]$", or (c) "$[\exists m \in M, |m| \leq 1, \bar{a} \cap \bar{b} = \emptyset]$";

if (b), then no map $m, [|m| > 1, \bar{a} \cap \bar{b} = \emptyset]$ exists (Contradiction); and

if (c), since $m, [|m| \leq 1, \bar{a} \cap \bar{b} = \emptyset]$, then two countries a, b do not exist in map m (Contradiction).

The argument for both the $\Rightarrow$ and $\Leftarrow$ cases of the present lemma are now complete. Hence, the present lemma holds (and establishes an existence proof of the concept of *mutual separation*). $\square$

Lem. 3.3. *Point P is a* corner *of map m:* $[\forall$*map* $m \in M, |m| > 2, \forall$*country* $t \in m$, *where* $B_t =$ *the outer boundary of country* $t, \exists$*point* $P \ni P \in B_t] \Leftrightarrow [P$ *is a* corner *of map* $m]$.

Proof. **Consider** the logical negation of the $\Leftarrow$ case of the definition of a corner of a collection of maps: $[\forall$map $m \in M, |m| > 2$, where $\forall$country $t \in m, B_t =$ the outer boundary of country $t, \exists$point $P \ni P \in B_t] \not\Leftarrow [P$ is a *corner* of map $m]$. (Def. 2.5) (The logical negation of the $\Leftarrow$ case of the present lemma)

Then "$\neg[\forall$map $m \in M, |m| > 2]$" (*i.e.*, either (a) "$[\exists m \in M, |m| \leq 2]$" is false, since the concept of "corner" requires more than two countries that belong to B_t (Contradiction); or (b) "$[\forall$country $t \in B_t, \exists$point $P \ni P \in B_t]$" is impossible, since every planar point, then, belongs to B_t) (Contradiction).

Consider now the statement "$[\forall$map $m \in M, |m| > 2, \forall$country $t \in m$, where $B_t =$ the outer boundary of country $t, \exists$point $P \ni P \in B_t] \Rightarrow [P$ is a *corner* of map $m]$", which, by Def. 2.5, satisfies the $\Rightarrow$ case of the present lemma. (Premise of the $\Rightarrow$ case of the present lemma)

Hence, since both the $\Leftarrow$ case and the $\Rightarrow$ case hold, the existence proof of the definition of the concept of the *corner* of map m is established. $\square$

4 Four-Colorability Theorem

Thm. 4.1.

The set of all four-colorable maps: $F = \{m \in M$ *is* four-colorable$\}$.

1. $\big[\forall m \ni |m| > 1, G$ *is a collection of ordered pairs of countries* $(u, v) \ni [u, v \in m] \ni |G| = n$, *where* n *does not exclude any possible transfinite number,*[5, 12] *and*
2. $\forall \eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, *each member of at least one ordered pair* $(a, b) \in h_\eta$ *is* mutually adjacent to *its pair-mate, and each member of every* ***remaining*** *ordered pair* $(a, b) \in G$ *is* mutually separated from *its pair-mate*$]\big] \Leftrightarrow [$*map* $m \in F]$.

Proof. Having established Lem. 3.1 (an existence proof for the concept of *adjacency*), Lem. 3.2 (an existence proof for the concept of *mutual separation*), and Lem. 3.3 (an existence proof for the concept of the *corner* of map m as a point P common to the closures of the boundaries of a collection B_t of three or more countries belonging to m), a proof of the present theorem will follow (where Lemmas 3.1 and 3.2 are incorporated explicitly, while Lem. 3.3 is implicitly incorporated).

Note: The proof of Thm. 4.1 involves Def. 2.3 (the concept of *adjacency*), Def. 2.4 (the concept of *mutual separation*), and Def. 2.5 (the concept of a *corner* of the countries in a map). Recall also that in 1890, P. J. Haewood proved that every map having five colors is four-colorable.[10]

Consider the logical negation of the $\Leftarrow$ case of the present theorem:

1. $\big[\forall m \ni |m| > 1, G$ is a collection of ordered pairs of countries $(u, v) \ni [u, v \in m] \ni |G| = n$, where n does not exclude any possible transfinite number,[5, 12] and
2. $\forall \eta \in \{1, 2, 3, 4\}, [h_\eta \subset G$, each member of at least one ordered pair $(a, b) \in h_\eta$ is *mutually adjacent to* its pair-mate, and each member of every **remaining** ordered pair $(a, b) \in G$ is *mutually separated from* its pair-mate$]\big] \not\Leftarrow [$map $m \in F]$. (Def. 2.1) (Def. 2.3) (Def. 2.4)

Then either

1. $[\nexists$map $m] \vee \neg[|m| > 1] \vee \big[[\nexists G$ a collection of ordered pairs $(u, v) \ni [u, v \in m]\big] \vee \neg[|G| = n]$,
2. or $\neg\big[\forall \eta \in \{1, 2, 3, 4\}, h_\eta \subset G$,
3. each [member of at least one ordered pair $(a, b) \in h_\eta$ is *mutually adjacent to* its pair-mate],
4. and [each member of every **remaining** ordered pair $(a, b) \in G$ is *mutually separated from* its pair-mate$]\big]$.

Consider the statements (a) "$[\nexists$map $m]$"; (b) $\vee$"$\neg[|m| > 1]$"; and (c) $\vee$"$\nexists G$ a collection of ordered pairs $(u, v) \ni [u, v \in m]$". (a) is false since map m then does not exist (Contradiction); (b) (stated as follows) $[|m| \leq 1]$ is false since m then includes only 1 country (Contradiction); and (c) is false since G then does not even exist (Contradiction).

The only remaining statements are as follows: "$\neg\big[\forall \eta \in \{1,2,3,4\}, [h_\eta \subset G$, each member of at least one ordered pair $(a,b) \in h_\eta$ is *mutually adjacent to* its pair-mate, and each member of every **remaining** ordered pair $(a,b) \in G$ is *mutually separated from* its pair-mate$]\big]$".

Then either

1. $\neg[\forall \eta \in \{1,2,3,4\}]$, so that either $\eta = 0$ or $\eta \geq 5$. The statement "$\eta = 0$" is nonsense, implying no ordered pair exists in G (Contradiction). The statement "$\eta \geq 5$" violates the Separation Axiom,[4, 7, 14] which states that "no map has five mutually adjacent countries" (Separation Axiom) (Contradiction).

2. $\neg[|G| = n]$ is false because, then, either $[n \leq 1] \vee [\exists n' \ni n' > n]$, so that either (a) "$n \leq 1$" implies no more than one ordered pair exists in G (Contradiction), or (b) "$[\exists n' \ni n' > n]$" is impossible, since no number exceeds n (*i.e.*, no number exceeds $|G|$: "$|G|$ does not exclude any possible transfinite number") (Contradiction).

A contradiction exists for every possibility under the $\Leftarrow$ case, completing the foregoing argument.

Now **consider** the $\Rightarrow$ case of the present theorem:

1. $\big[\forall m \ni |m| > 1, G$ is a collection of ordered pairs of countries $(u,v), [u,v \in m]$,

2. $\eta \in \{1,2,3,4\}, [h_\eta \subset G$, each member of at least one ordered pair $(a,b) \in h_\eta$ is *mutually adjacent to* its pair-mate, and each member of every **remaining** ordered pair $(a,b) \in G$ is *mutually separated from* its pair-mate$]\big] \Rightarrow [\text{map } m \in F]$.

Then the statement "[map $m \in F$]", and therefore the statement "[m is four-colorable]", satisfies the premise of the $\Rightarrow$ case of the present theorem, and thus the $\Rightarrow$ case holds. (Def. 2.7) (Thm. 4.1) (Premise of the $\Rightarrow$ case of the present theorem)

Since both the $\Leftarrow$ case and the $\Rightarrow$ case hold, then the present theorem holds. $\square$

5 Crucial Definitions

The crucial definitions in this paper are as follows: the *concept of adjacency* (Def. 2.3); the *concept of mutual separation* (Def. 2.4); the idea of a *corner of the countries in a map* (*i.e.*, a point P common to the closures of the boundaries of a collection B_t of three or more countries belonging to the map) (Def. 2.5); and the *four-colorability of maps* (Def. 2.7).

(Another important concept that is defined but nevertheless disregarded in this paper is that of an *isolated point* (Def. 2.6)). (See Footnote 6.)

6 Set-Builder Notation

Other notational systems exist (*e.g.*, those based on model theory).[11] This paper adopted the Set-Builder Notation,[13] because of its accuracy, completeness, effectiveness, and universality (see documentation for the TeXShop™ class files).

7 Discussion

Following a brief literature review, this paper presented proofs of adjacency-existence (Lem. 3.1), mutual-separation-existence (Lem. 3.2), and the existence of a corner of the countries in a map (Lem. 3.3). Finally, this paper established Thm. 4.1: every map is four-colorable.

The following list indicates, for the three lemmas and the theorem, the items (lemmas, definitions, contradictions, *etc.*) that were relied upon in their respective proofs:

1. The proof of Lem. 3.1 relied upon Def. 2.1, Def. 2.2, Def. 2.3, Premise of the $\Rightarrow$ case, and Contradiction.

2. The proof of Lem. 3.2 relied upon Def. 2.1, Def. 2.2, Def. 2.4, Premise of the $\Rightarrow$ case, and Contradiction.

3. The proof of Lem. 3.3 relied upon Def. 2.1, Def. 2.2, Def. 2.4, Def. 2.5, Premise of the $\Rightarrow$ case, and Contradiction.

4. The proof of Thm. 4.1 relied upon Def. 2.1, Def. 2.3, Def. 2.4, Def. 2.5, Def. 2.7, Set-Builder Notation logic, References,[5, 12] Contradiction, and Lem. 3.1, Lem. 3.2, (implicitly) Lem. 3.3, and the Separation Axiom.[4, 7, 14]

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